15.5 Directional derivative

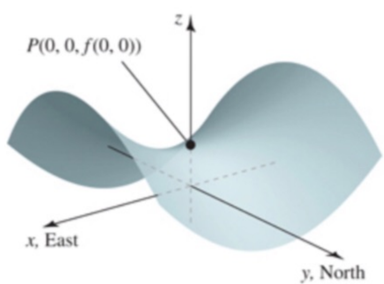
DEFINITION Directional Derivative

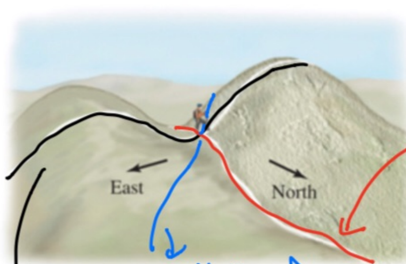
Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}}f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

What is the differenc between the partial dev and directional dev.?





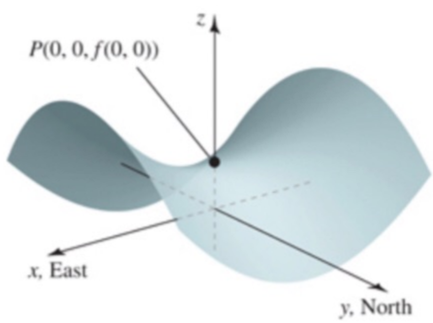
fy north/south

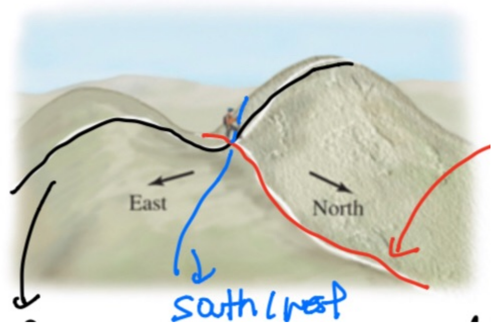
south/west

fx east-west

tells difference of height as the hiker moves

How about in the northeast/southwest direction?





An:

that is given by <u>the directional dev</u>.

change of $f(x,y)$ in some direction $\vec{u}$

where $\vec{u}$ is a <u>unit vector</u> giving us

the direction.

We'll start with the gradient of a function first.

DEFINITION Gradient (Two Dimensions)

Let f be differentiable at the point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

ex) comput $\nabla f(1.2)$ where $f(x,y) = 2xy + x^2$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2y+2x, 2x \rangle$

$\nabla f(1.2) = \langle 2(2)+2(1), 2(1) \rangle = \langle 6.2 \rangle$

Sometimes you'll be given a vector make sure you change it to the corresponding unit vector!

THEOREM 15.10 Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}} f(a,b) = \underline{\langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle}.$$

↳ $\nabla f(a,b) \cdot \vec{u}$ dot product of gradient of f at (a,b) and unit vector $\vec{u}$.

ex) Compute the directional dev. of $f(x,y) = \sqrt{x^2+y^2+2}$ at (1,2) in the direction of <2,2>.

$$\nabla f(x,y) = \left\langle \frac{2x}{2\sqrt{x^2+y^2+2}}, \frac{2y}{2\sqrt{x^2+y^2+2}} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+2}}, \frac{y}{\sqrt{x^2+y^2+2}} \right\rangle$$

$$\nabla f(1,2) = \left\langle \frac{1}{\sqrt{1+4+2}}, \frac{2}{\sqrt{1+4+2}} \right\rangle = \left\langle \frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}} \right\rangle$$

continue ...

· continue

direction $\langle 2,2\rangle$ is not a unit vector.

a unit vector in the same direction is obtained

by $\dfrac{\langle 2,2\rangle}{|\langle 2,2\rangle|} = \dfrac{\langle 2.2\rangle}{\sqrt{2^2+2^2}} = \dfrac{1}{\sqrt{8}}\langle 2.2\rangle$

$\vec{u} = \left\langle \dfrac{2}{\sqrt{8}}, \dfrac{2}{\sqrt{8}} \right\rangle$

the directional vector of f at pt (1.2)

in the direction of $\langle 2,2\rangle$ is

$\nabla f(1.2)\cdot u = \left\langle \dfrac{1}{\sqrt{7}}, \dfrac{2}{\sqrt{7}} \right\rangle \bullet \left\langle \dfrac{2}{\sqrt{8}}, \dfrac{2}{\sqrt{8}} \right\rangle$

dot product.

$= \dfrac{2}{\sqrt{56}} + \dfrac{4}{\sqrt{56}} = \boxed{\dfrac{6}{\sqrt{56}}}$

THEOREM 15.11 Directions of Change
Let f be differentiable at (a, b) with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $|\nabla f(a, b)|$.
2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-|\nabla f(a, b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

On HW, you are ask to find the "unit vector" in the direction of sharpest increase / decrease. In the theorem, $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$ might not be a unit vector. make sure you change the vector $\nabla f(a,b)$ into a unit vector.

ex) $f(x,y) = 4\ln(x+y)$ $P(2,3)$

a) Find a unit vector that gives the direction of sharpest increase / decrease.

$$\nabla f(x,y) = \left\langle \frac{4}{x+y}, \frac{4}{x+y} \right\rangle$$

$$\nabla f(2,3) = \left\langle \frac{4}{2+3}, \frac{4}{2+3} \right\rangle = \left\langle \frac{4}{5}, \frac{4}{5} \right\rangle$$

Note that $\langle 4/5, 4/5 \rangle$ is not a unit vector.

So the answer should be $\dfrac{\langle 4/5, 4/5 \rangle}{|\langle 4/5, 4/5 \rangle|} = \dfrac{1}{\sqrt{16/25+16/25}} \langle 4/5, 4/5 \rangle$

$$= \frac{1}{\sqrt{32/25}} \langle 4/5, 4/5 \rangle = \sqrt{\frac{25}{32}} \left\langle \frac{4}{5}, \frac{4}{5} \right\rangle$$

$$= \left\langle \frac{4}{\sqrt{32}}, \frac{4}{\sqrt{32}} \right\rangle$$

max decrease $\longrightarrow$ $-\left\langle \frac{4}{\sqrt{32}}, \frac{4}{\sqrt{32}} \right\rangle = \left\langle -\frac{4}{\sqrt{32}}, -\frac{4}{\sqrt{32}} \right\rangle$

b) Find a vector that points in the direction of no change.

do not have to be a unit vector

$\langle -\frac{4}{5}, \frac{4}{5} \rangle$ or $\langle \frac{4}{5}, \frac{4}{5} \rangle$ or $\langle -4, 4 \rangle$ or $\langle 4, 4 \rangle \cdots$

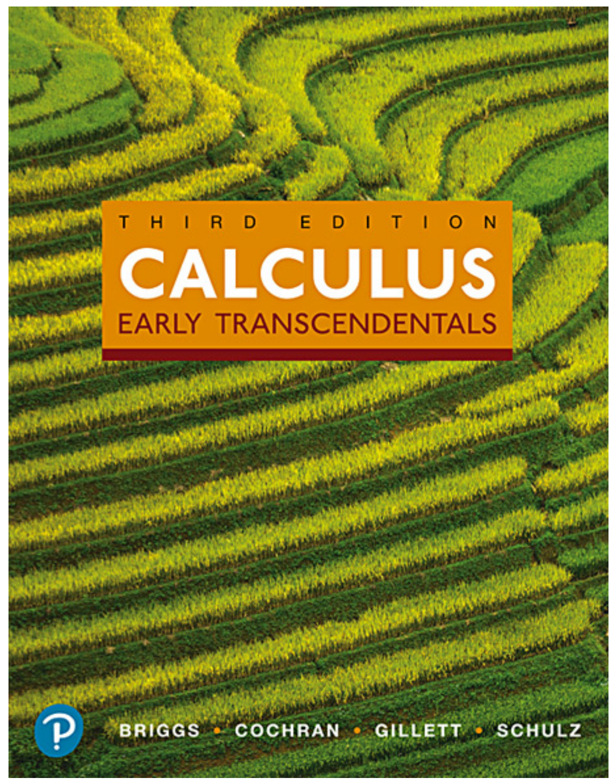
any vector $\vec{u}$ such that

$\vec{u} \cdot \langle \frac{4}{5}, \frac{4}{5} \rangle = 0$ works.

dot product.

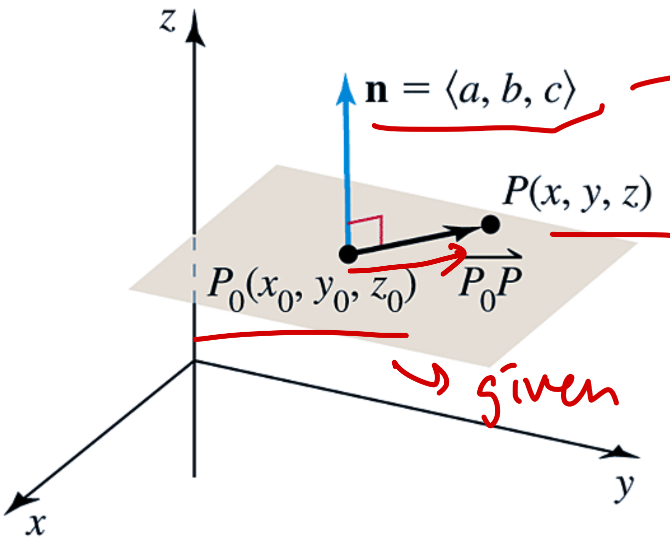
13.5

Planes in Space



DEFINITION **Plane in $\mathbb{R}^3$**

Given a fixed point P_0 and a nonzero **normal vector n**, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to **n** is called a **plane** (Figure 13.72).



→ $\vec{n}$ is the normal vector

→ any point on plane.

$\overrightarrow{P_0P} \perp \vec{n}$.

The orientation of a plane is specified by a normal vector **n**. All vectors $\overrightarrow{P_0P}$ in the plane are orthogonal to **n**.

General Equation of a Plane in $\mathbb{R}^3$

The plane passing through the point $P_0(x_0, y_0, z_0)$ with a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ is described by the equation

components of $\vec{n}$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

simplify

e.g. • $P_0(\underset{x_0}{2}, \underset{y_0}{-3}, \underset{z_0}{0})$.

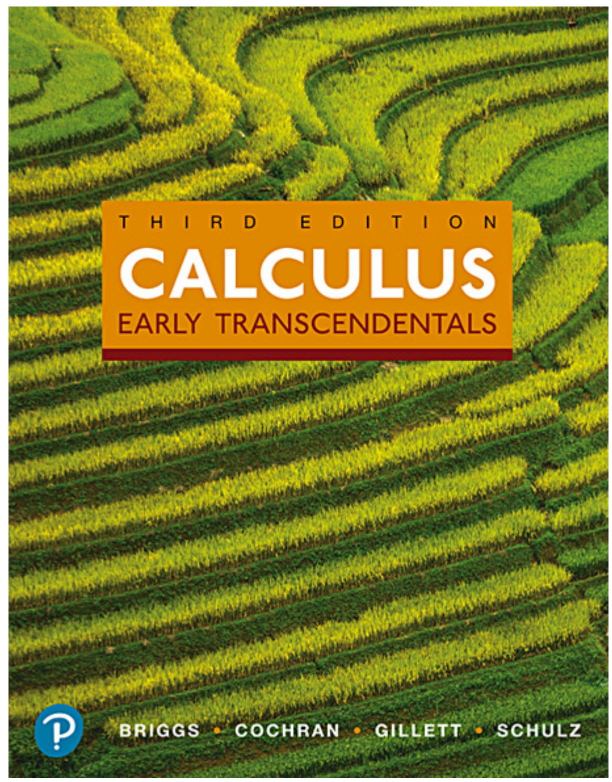
• $\vec{n} = \langle -\frac{1}{3}, 0, 2 \rangle$ What is the equation of the plane?

$$-\frac{1}{3}(x - 2) + 0(y - (-3)) + 2(z - 0) = 0$$

$$-\frac{1}{3}x + \frac{2}{3} + 2z = 0, \quad \boxed{-\frac{1}{3}x + 2z = -\frac{2}{3}}$$

15.6

Tangent Planes

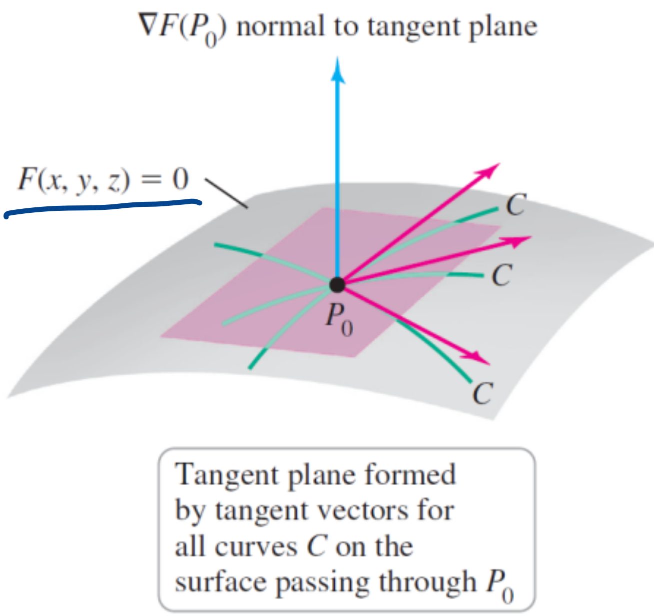


DEFINITION **Equation of the Tangent Plane for $\boxed{F(x, y, z) = 0}$**

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The plane tangent to the surface $F(x, y, z) = 0$ at P_0, called the **tangent plane**, is the plane passing through P_0 orthogonal to $\nabla F(a, b, c)$. An equation of the tangent plane is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

gradient of F = normal vector of tangent plane.



Timeline

Example - Equation of a tangent plane for $F(x,y,z)=0$

$x^2+y^3+z^4=2$ at the points $(1,0,1)$ and $(-1,1,0)$.

$F(x,y,z)=x^2+y^3+z^4$

$\nabla F=\langle F_x, F_y, F_z\rangle$

$P_0(1,0,1)$

$=\langle 2x, 3y^2, 4z^3\rangle$

$\nabla F(1,0,1)=\langle 2,0,4\rangle=\vec{n}$

$2(x-1)+0(y-0)+4(z-1)=0$

$2x-2+4z-4=0$

$2x+4z=6$

$\nabla F(-1,1,0)=\langle -2,3,0\rangle=\vec{n}$

$-2(x-(-1))+3(y-1)+0(z-0)=0$

$-2x-2+3y-3=0$

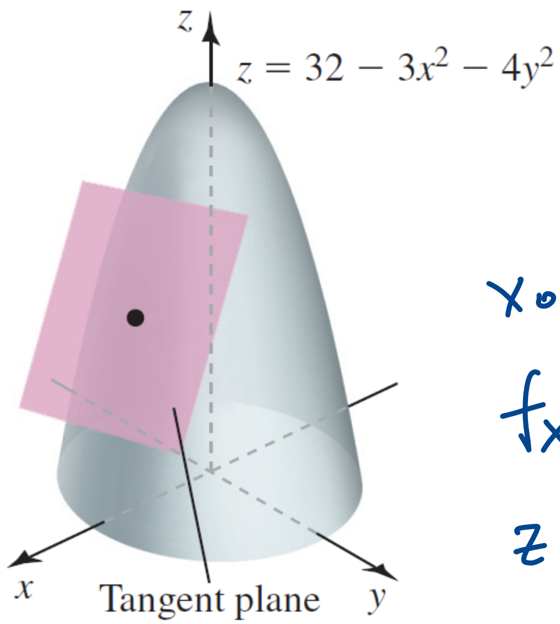
$-2x+3y=5$

$F(x,y,z)=0$ $\quad F_x(x-x_0)+F_y(y-y_0)+F_z(z-z_0)=0.$

Tangent Plane for $z = f(x, y)$

Let f be differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$



$\underbrace{f(x,y)-z}_{F(x,y,z)}=0.$ $\quad \vec{\nabla}F=\langle f_x, f_y, -1\rangle$

$x_0, y_0 \Rightarrow z_0 = f(x_0, y_0) = C$

$f_x(x-x_0)+f_y(y-y_0)-1(z-C)=0$

$z = f_x(x-x_0)+f_y(y-y_0)+C$

$= f_x(x-x_0)+f_y(y-y_0)+f(x_0,y_0)$

Example - Equation of a tangent plane for $z = f(x, y)$ $= \sin(xy) + 2$.

$z = \sin xy + 2$ at the points (1,0,2) and (0,5,2).

$a = 1 \quad b = 0. \quad c = 2$

$z = f_x(x-1) + f_y(y-0) + 2$

$f_x = \cos(xy) \cdot y \quad f_y = \cos(x,y) \cdot x$

$f_x(1.0) = \cos(0) \cdot 0 = 0. \quad f_y(1.0) = \cos(0) \cdot 1 = 1$

$z = 0(x-1) + 1(y) + 2 = y + 2.$ $\vec{n} = \langle 0, 1, -1 \rangle = \langle f_x, f_y, -1 \rangle$

$z = f_x(x-0) + f_y(y-5) + 2.$

$f_x(0.5) = 1.5 \quad f_y(0,5) = 0.$ $\vec{n} = \langle 5, 0, -1 \rangle$

$z = 5x + 2$

Find an eq of the plane tangent to the given surface at the given pt.

$z = \cos(xy)$; $(1, \frac{\pi}{2}, 0)$.

$(x_0, y_0, f(x_0, y_0))$

$z = \cos(1 \cdot \frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$\boxed{z = f(x, y)}$

$F(x, y, z) = \cos(xy) - z = 0$

$\nabla F(x, y, z) = \langle \underbrace{-\sin(xy) \cdot y}_{f_x}, \underbrace{-\sin(xy) \cdot x}_{f_y}, -1 \rangle$

$\nabla F(1, \frac{\pi}{2}, 0) = \langle -\sin(\frac{\pi}{2}) \cdot \frac{\pi}{2}, -\sin(\frac{\pi}{2}) \cdot 1, -1 \rangle$

$= \langle -\frac{\pi}{2}, -1, -1 \rangle$

$-\frac{\pi}{2}(x - 1) - 1(y - \frac{\pi}{2}) - 1(z - 0) = 0$

$-\frac{\pi}{2}x + \frac{\pi}{2} - y + \frac{\pi}{2} - z = 0$

$\boxed{-\frac{\pi}{2}x - y - z = -\pi}$

$z = f_x(x - x_0) + f_y(y - y_0) + f(x_0, y_0)$

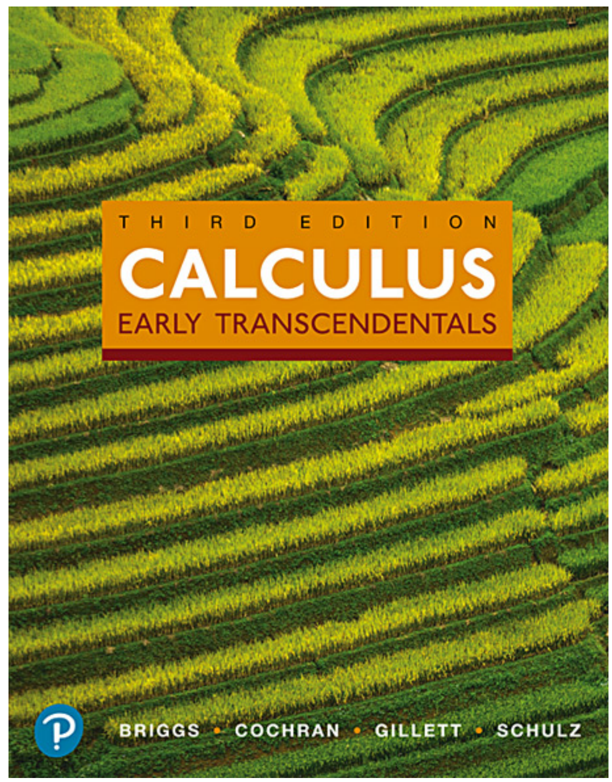
$= -\frac{\pi}{2}(x-1) - 1(y - \frac{\pi}{2}) + 0$

$= -\frac{\pi}{2}x + \frac{\pi}{2} - y + \frac{\pi}{2}$

$\boxed{z = -\frac{\pi}{2}x - y + \pi}$

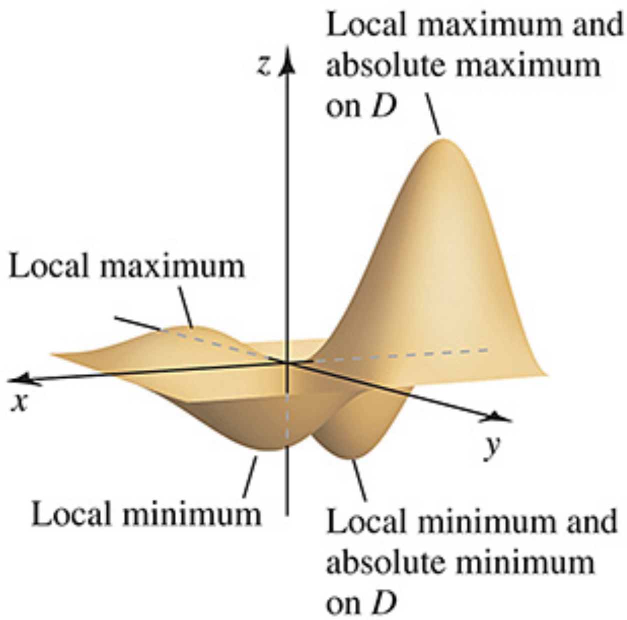
15.7

Maximum/Minimum Problems



DEFINITION **Local Maximum/Minimum Values**

Suppose (a, b) is a point in a region R on which f is defined. If $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local maximum value** of f. If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f and in some open disk centered at (a, b), then $f(a, b)$ is a **local minimum value** of f. Local maximum and local minimum values are also called **local extreme values** or **local extrema**.



THEOREM 15.14 **Derivatives and Local Maximum/Minimum Values**

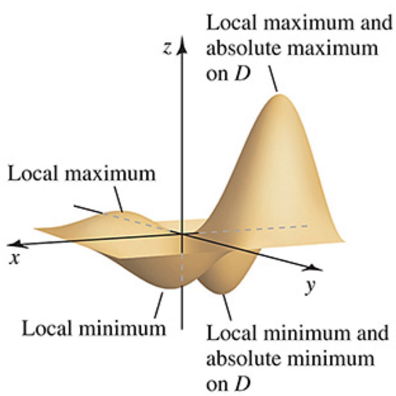
If f has a local maximum or minimum value at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.

DEFINITION **Critical Point**

An interior point (a, b) in the domain of f is a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$, or

2. at least one of the partial derivatives f_x and f_y does not exist at (a, b).



Example: Find critical points

$$f(x, y) = x^3 + 6xy - 6x + y^2 - 2y$$

$f_x = 3x^2 + 6y - 6 = 0$

$3x^2 = -6y + 6$

$x^2 = \underbrace{-2y + 2}$

$f_y = 6x + 2y - 2 = 0$

$6x = \underbrace{-2y + 2}$

$x^2 = 6x$

$x^2 - 6x = 0$

$x(x-6) = 0$

$x = 0, 6$

$\underline{x = 0}$

$-2y + 2 = 0$

$2 = 2y$

$1 = y$

$\underline{x = 6}$

$-2y + 2 = 36$

$-2y = 34$

$y = -17$

Critical Points: $(0, 1)$ $(6, -17)$

e.g.) Find all critical pts of $f(x,y) = -4xy - x^4 - y^4$

$f_x = -4y - 4x^3$ $\qquad$ $f_y = -4x - 4y^3$ $\qquad$ we need both $f_x = f_y = 0$

so, $-4y - 4x^3 = 0$ $\qquad$ $-4x - 4y^3 = 0$

$\hookrightarrow -4y = 4x^3$

& $y = -x^3$

$-4x - 4(x^3)^3 = 0$

$-x + x^9 = 0$

$x(1 - x^8) = 0$ $\qquad$ either $x = 0$ or $x = \pm 1$.

critical points:

if $x = 0 \Rightarrow y = 0$ $\qquad$ $(0,0)$

$x = 1 \Rightarrow y = -1$ $\qquad$ $(1,-1)$

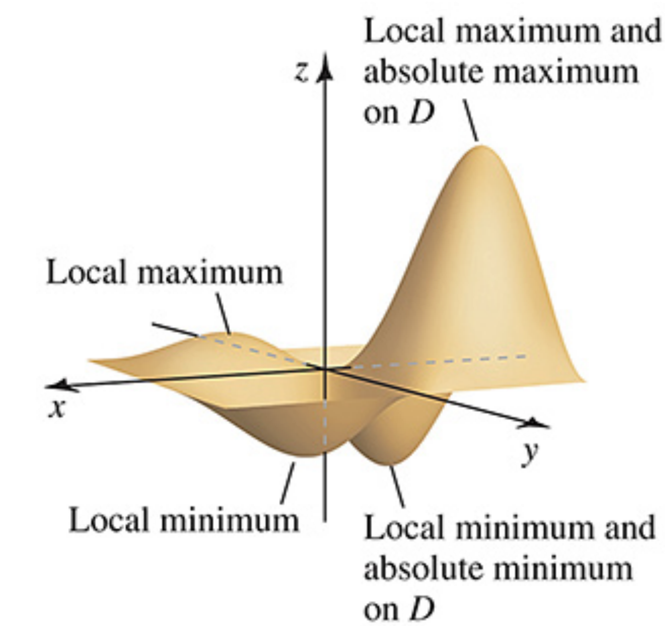
$x = -1 \Rightarrow y = 1$ $\qquad$ $(-1,1)$

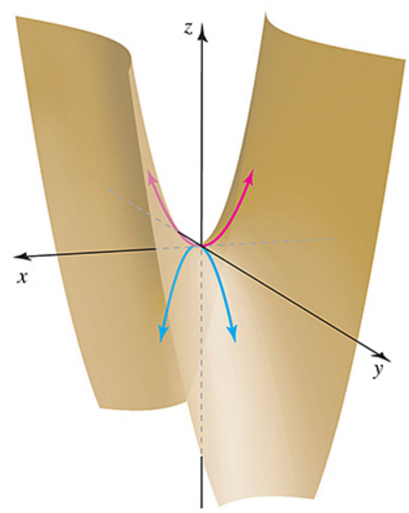
In this section, you'll be using a lot of algebra.

If you are having trouble with it, see me or math department tutors.

Second derivative Test:

Second partial derivative f_{xx} f_{yy} f_{xy} f_{yx}

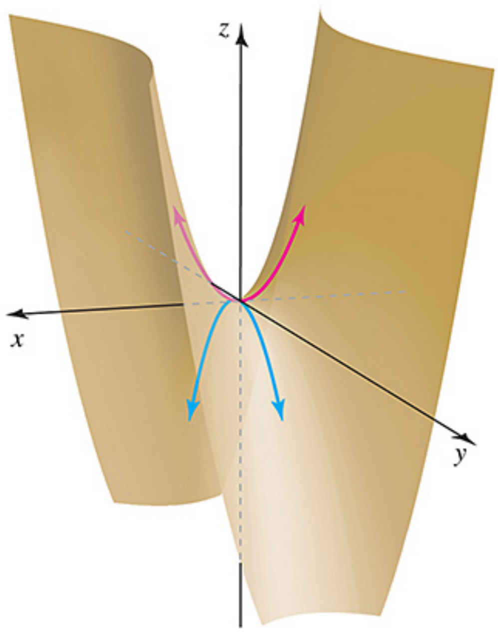




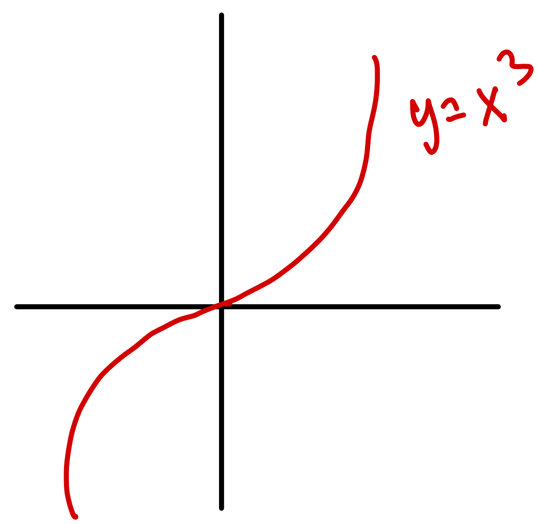
The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).

DEFINITION **Saddle Point**

Consider a function f that is differentiable at a critical point (a, b). Then f has a **saddle point** at (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.



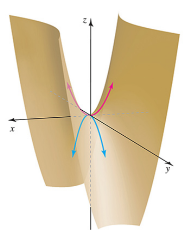
The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at $(0, 0)$.



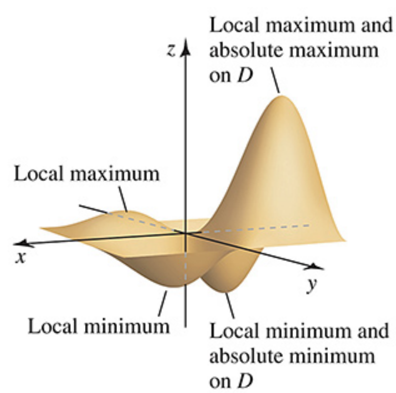
THEOREM 15.15 **Second Derivative Test**

Suppose the second partial derivatives of f are continuous throughout an open disk centered at the point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - (f_{xy}(x, y))^2$. ← Discriminant

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).
4. If $D(a, b) = 0$, then the test is inconclusive.



The hyperbolic paraboloid $z = x^2 - y^2$ has a saddle point at (0, 0).



Example: Find and determine the types of critical points

$f(x, y) = x^4 + 2y^2 - 4xy$

$f_x = 4x^3 - 4y = 0$

$4x^3 = 4y$

$x^3 = y$

$f_y = 4y - 4x = 0$ defined everywhere

$4y = 4x$

$y = x$

$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x+1)(x-1) = 0$

$x = 0, -1, 1.$

$x = 0 \Rightarrow y = 0$

$x = -1 \Rightarrow y = -1$

$x = 1 \Rightarrow y = 1$

critical point

$(0, 0)$

$(-1, -1)$

$(1, 1)$

Example: Find and determine the types of critical points

$f(x, y) = x^4 + 2y^2 - 4xy$

$f_x = 4x^3 - 4y$, $f_y = 4y - 4x$, Critical Points $(0,0)$, $(1,1)$ $(-1,-1)$

$f_{xx} = 12x^2$ $\quad f_{xy} = -4 \quad f_{yy} = 4$

$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 12x^2 \cdot 4 - (-4)^2 = 48x^2 - 16$

$D(0,0) = 48 \cdot 0^2 - 16 = -16 < 0 \Rightarrow$ saddle point at $(0,0)$

$D(1,1) = 48 \cdot 1^2 - 16 = 32 > 0$ & $f_{xx}(1,1) = 12 \cdot 1^2 = 12 > 0$

$\Rightarrow$ local min at $(1,1)$

$D(-1,-1) = 48(-1)^2 - 16 = 32 > 0$ & $f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0$

$\Rightarrow$ local min at $(-1,-1)$

Use the 2nd derivative test to classify the critical pts of the function $f(x,y) = -4xy - x^4 - y^4$.

Critical pts: $(-1,1)$, $(1,-1)$, $(0,0)$

Partial derivatives: $f_x = -4y - 4x^3$ $f_y = -4x - 4y^3$ $f_{xy} = -4$

$f_{xx} = -12x^2$ $f_{yy} = -12y^2$

$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (-12x^2)(-12y^2) - (-4)^2$

$= 144x^2y^2 - 16$

$D(0,0) = 144 \cdot 0^2 \cdot 0^2 - 16 = -16 < 0$ saddle pt at $(0,0)$

$D(-1,1) = 144(-1)^2(1)^2 - 16 > 0$, $f_{xx}(-1,1) = -12(-1)^2 < 0$, local max at $(-1,1)$

$D(1,-1) = 144(1)^2(-1) - 16 > 0$, $f_{xx}(1,-1) = -12(1)^2 < 0$, local max at $(1,-1)$

2nd derivative test is conclusive for all critical points.

Example: Find and determine the types of critical points

$f(x, y) = 4 + x^4 + 3y^4$

Find $f_x = 4x^3 = 0$ $\quad f_y = 12y^3 = 0$ $\quad$ critical point: $(0,0)$

$x = 0$ $\quad y = 0$

$f_{xx} = 12x^2$ $\quad f_{xy} = 0$ $\quad f_{yy} = 36y^2$

$D(x,y) = (12x^2)(36y^2) - 0^2 = 432x^2y^2$

$D(0,0) = 0 \Rightarrow$ Inconclusive

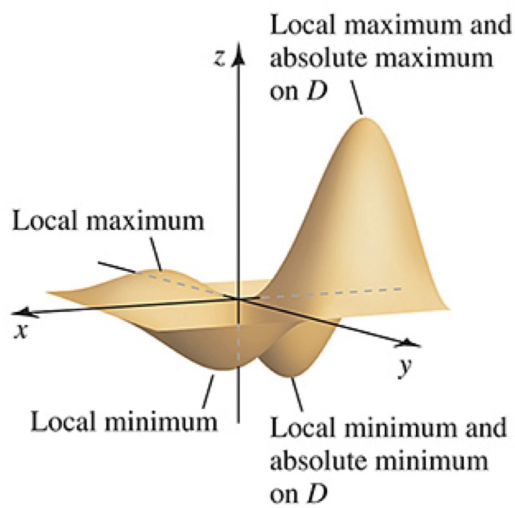
D does not tell use the types

Since $f(x,y) > 4$ for all $(x,y) \neq (0,0)$ and $f(0,0) = 4$

$(0,0)$ is a local and absolute minimum of the function f

DEFINITION Absolute Maximum / Minimum Values

Let f be defined on a set R in $\mathbb{R}^2$ containing the point (a, b). If $f(a, b) \geq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute maximum value** of f on R. If $f(a, b) \leq f(x, y)$ for every (x, y) in R, then $f(a, b)$ is an **absolute minimum value** of f on R.



PROCEDURE **Finding Absolute Maximum / Minimum Values on Closed Bounded Sets**

Let f be continuous on a closed bounded set R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Determine the values of f at all critical points in R.
2. Find the maximum and minimum values of f on the boundary of R.
3. The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.

Example: Extreme values on a closed region

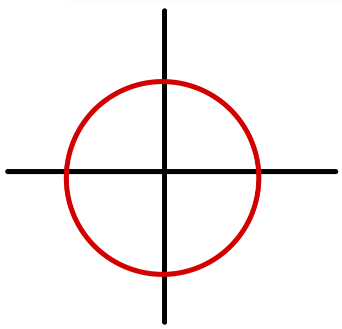
$$f(x,y) = 2x^2 + y^2; R = \{(x,y) : x^2 + y^2 \leq 16\}$$

Inside: $f_x = 4x = 0 \Rightarrow x = 0$

defined everywhere $f_y = 2y = 0 \Rightarrow y = 0$

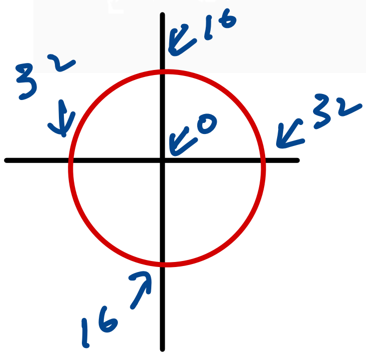
critical pt: $(0,0)$

$f(0.0) = 2 \cdot 0 + 0 = 2$



Example: Extreme values on a closed region

$$f(x, y) = 2x^2 + y^2; R = \{(x, y) : x^2 + y^2 \le 16\}$$



On the boundary: $x = 4\cos t$, $y = 4\sin t$

$0 \le t \le 2\pi$

$$f(t) = 2(4\cos t)^2 + (4\sin t)^2$$
$$= 32\cos^2 t + 16\sin^2 t$$

$$\frac{df}{dt} = -64\cos t \sin t + 32\sin t\cos t = -32\cos t\sin t = 0$$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

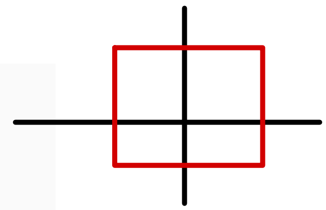
$f(0) = 32$, $f(\frac{\pi}{2}) = 16$, $f(\pi) = 32$, $f(\frac{3\pi}{2}) = 16$

Points	Value
(0,0)	0
(4,0)	32
(0,4)	16
(-4,0)	32
(0,-4)	16

min

max

Example: Extreme values on an open region



$f(x, y) = x^2 - y^2; R = \{(x, y) : |x| < 1, |y| < 1\}$

Inside: $f_x = 2x = 0 \Rightarrow x = 0$

$f_y = -2y = 0 \Rightarrow y = 0$

critical point $(0, 0)$

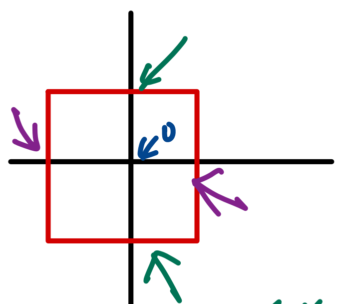
on the boundary (not included).

$y = 1, \ -1 \le x \le 1$

$f(x) = x^2 - 1$

$x = -1, -1 \le y \le 1$

$f(y) = 1 - y^2$



$x = 1, \ -1 \le y \le 1$

$f(y) = 1 - y^2$

$y = -1, \ -1 \le x \le 1$

$f(x) = x^2 - 1$

Example: Extreme values on an open region

$f(x,y) = x^2 - y^2; R = \{(x,y) : |x| < 1, |y| < 1\}$

On the boundary

Top + Bottom $y = \pm 1$

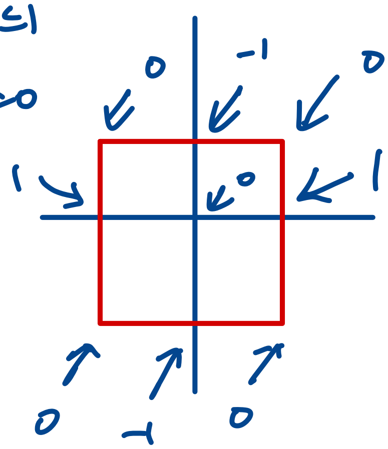
$f(x) = x^2 - 1, \ -1 \le x \le 1$

$f'(x) = 2x = 0 \rightarrow x = 0$

$f(-1) = 0$

$f(0) = -1$

$f(1) = 0$



Left + Right $x = \pm 1$

$f(y) = 1 - y^2, \ 1 \le y \le 1$

$f'(y) = -2y = 0 \rightarrow y = 0$

$f(-1) = 0$

$f(0) = 1$

$f(1) = 0$

No absolute maximum or minimum on open R.

Yes on $\bar{R}$. max=1 min=-1

15.8

Lagrange Multipliers

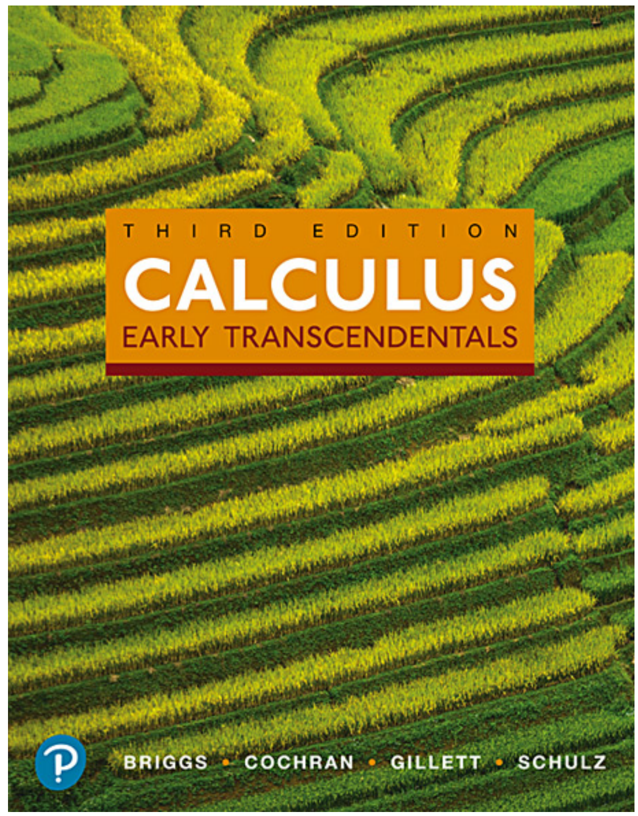
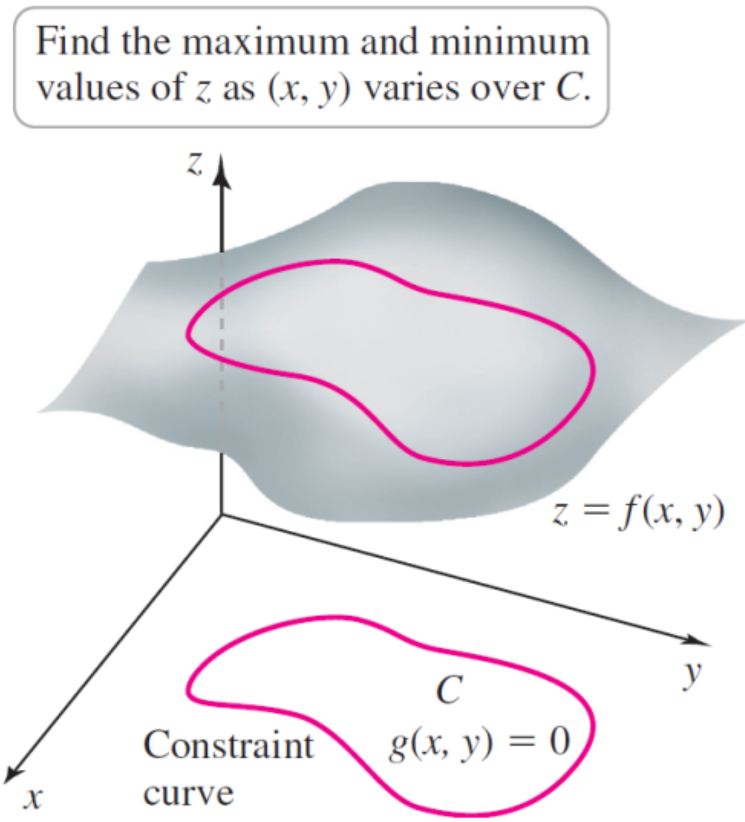


Figure 15.79


Find the maximum and minimum values of z as (x, y) varies over C.
z
$z = f(x, y)$
y
C
$g(x, y) = 0$
Constraint curve
x

PROCEDURE **Lagrange Multipliers: Absolute Extrema on Closed and Bounded Constraint Curves**

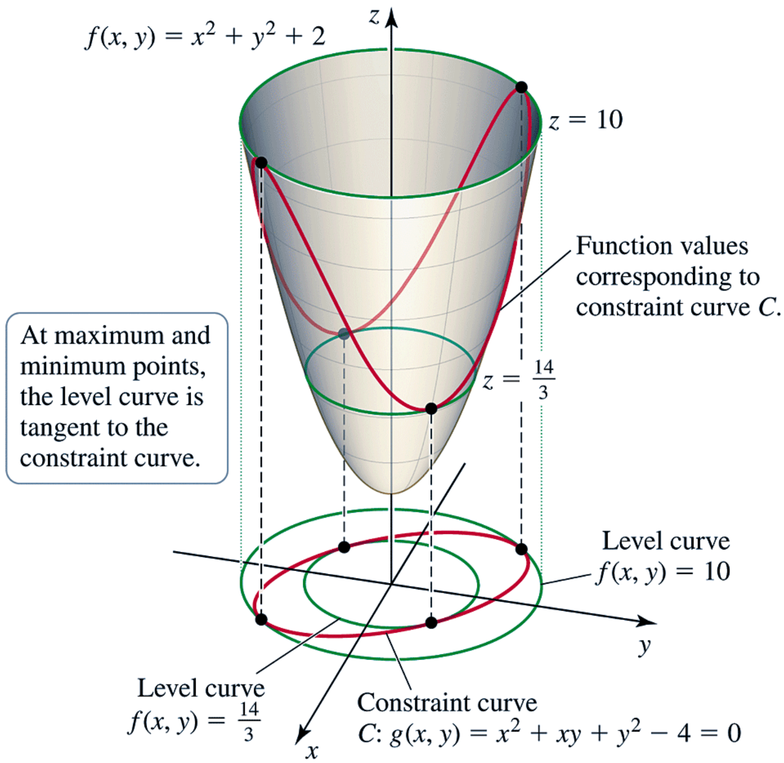
Let the objective function f and the constraint function g be differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x, y) \neq \mathbf{0}$ on the curve $g(x, y) = 0$. To locate the absolute maximum and minimum values of f subject to the constraint $g(x, y) = 0$, carry out the following steps.

1. Find the values of x, y, and λ (if they exist) that satisfy the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$

2. Evaluate f at the values (x, y) found in Step 1 and at the endpoints of the constraint curve (if they exist). Select the largest and smallest corresponding function values. These values are the absolute maximum and minimum values of f subject to the constraint.

Figure 15.81 (a & b)



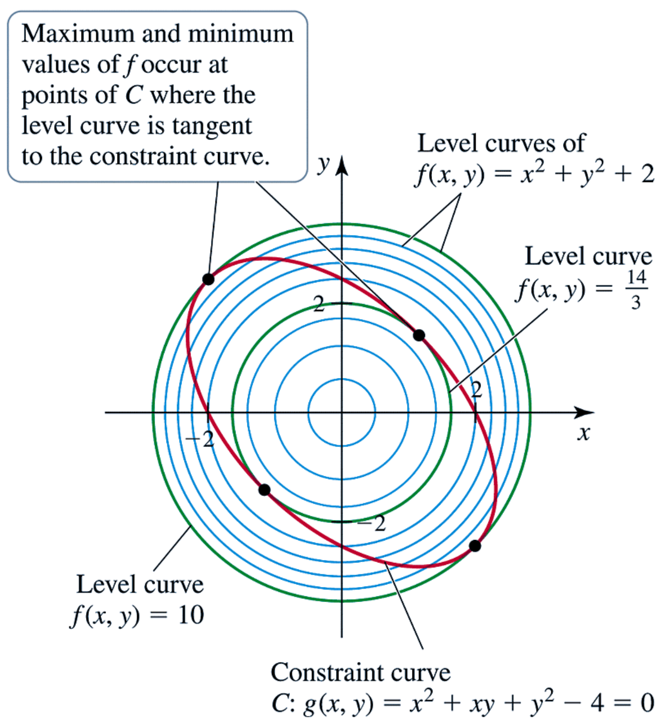
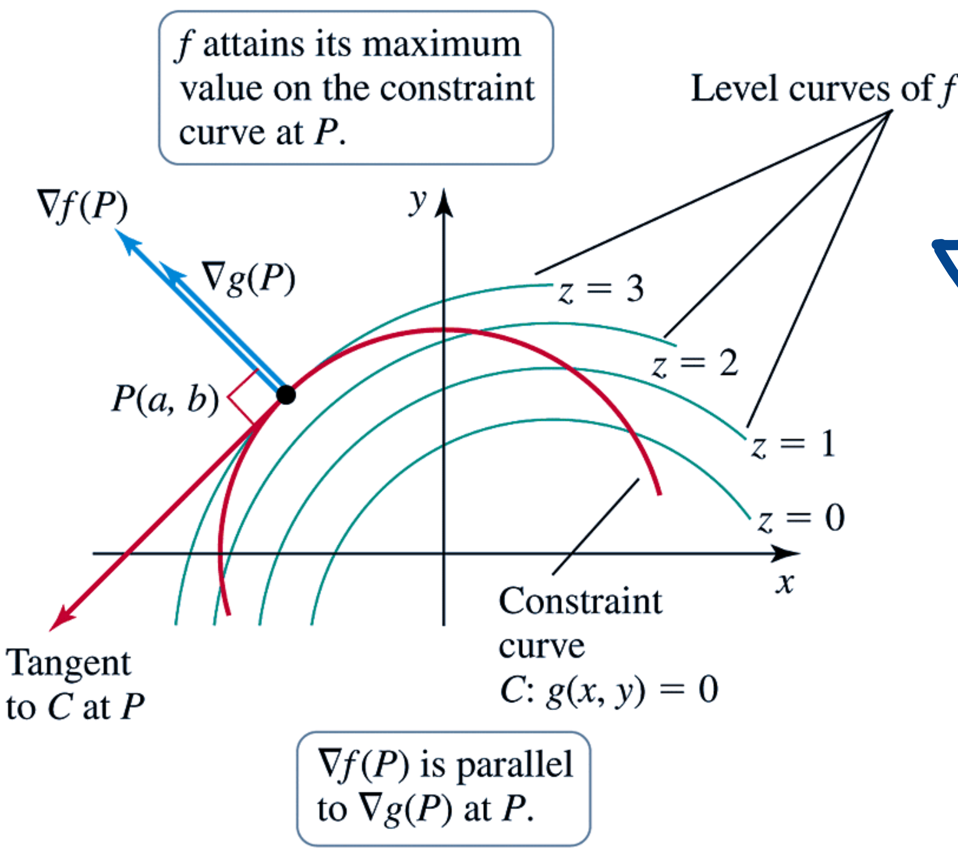


Figure 15.80



$\nabla f = \lambda \nabla g$

Example

Use Lagrange multipliers to find the absolute maximum and minimum of the function $f(x,y)=2x^2+y^2$ subject to the constraint $\underline{x^2+2y+y^2=15}$

$$g(x,y)=x^2+2y+y^2-15$$

$$\vec{\nabla} f(x,y)=\langle 4x, 2y\rangle \qquad \vec{\nabla} g(x,y)=\langle 2x, 2+2y\rangle$$

$$\vec{\nabla} f=\lambda\vec{\nabla} g \Rightarrow \langle 4x,2y\rangle=\lambda\langle 2x, 2+2y\rangle$$

$$\underline{4x=\lambda 2x} \quad \text{and} \quad \underline{2y=\lambda(2+2y)}$$

$$4x-\lambda 2x=0$$

$$2x(2-\lambda)=0 \Rightarrow x=0 \quad \text{or } \lambda=2$$

$\underline{x=0}$: $x^2+2y+y^2-15=0 \Rightarrow y^2+2y-15=0 \Rightarrow y=-5,\ y=3$

Pts: $(0,-5)$ $(0,3)$

$\underline{\lambda=2}$: $2y=\lambda(2+2y) \Rightarrow 2y=4+4y \Rightarrow -2y=4 \Rightarrow y=-2$

$x^2+2(-2)+(-2)^2-15=0 \Rightarrow x^2=15 \Rightarrow x=\pm\sqrt{15}$ Pts: $(\sqrt{15},-2)$, $(-\sqrt{15},-2)$

(x,y)	$f(x,y)=2x^2+y^2.$	
$(0;5)$	25	
$(0,3)$	9	Minimum Value
$(\sqrt{15},-2)$	34	maximum Value.
$(-\sqrt{15},-2)$	34	

Example

Use Lagrange multipliers to find the absolute maximum and minimum of the function $f(x,y,z) = x+y+z$ subject to the constraint $x^2+y^2+z^2-xy=5$

$f(x,y,z) = x+y+z$ $\qquad$ $g(x,y,z) = x^2+y^2+z^2-xy-5$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$

$1 = \lambda(2x-y) \qquad 1 = \lambda(2y-x) \qquad 1 = \lambda(2z)$

$\lambda(2x-y) = \lambda(2y-x) \qquad (\lambda \neq 0)$

$2x-y = 2y-x$

$3x = 3y$

$x = y$

$x=y$: $1 = \lambda(2x-y) \Rightarrow 1 = \lambda x$

$\lambda(2z) = \lambda x \Rightarrow 2z = x \Rightarrow z = \frac{1}{2}x$

$x^2 + x^2 + (\frac{1}{2}x)^2 - x^2 - 5 = 0.$

$\Rightarrow x^2 + \frac{1}{4}x^2 = 5$

$\Rightarrow \frac{5}{4}x^2 = 5 \Rightarrow x = \pm 2$

$x = y$, $z = \frac{1}{2}x$, $x = \pm z$

(x, y, z)	$f(x, y, z) = x+y+z.$	
$(2, 2, 1)$	5	$\leftarrow$ maximum
$(-2, -2, -1)$	-5	$\leftarrow$ minimum.

Figure 15.82 (c & d)

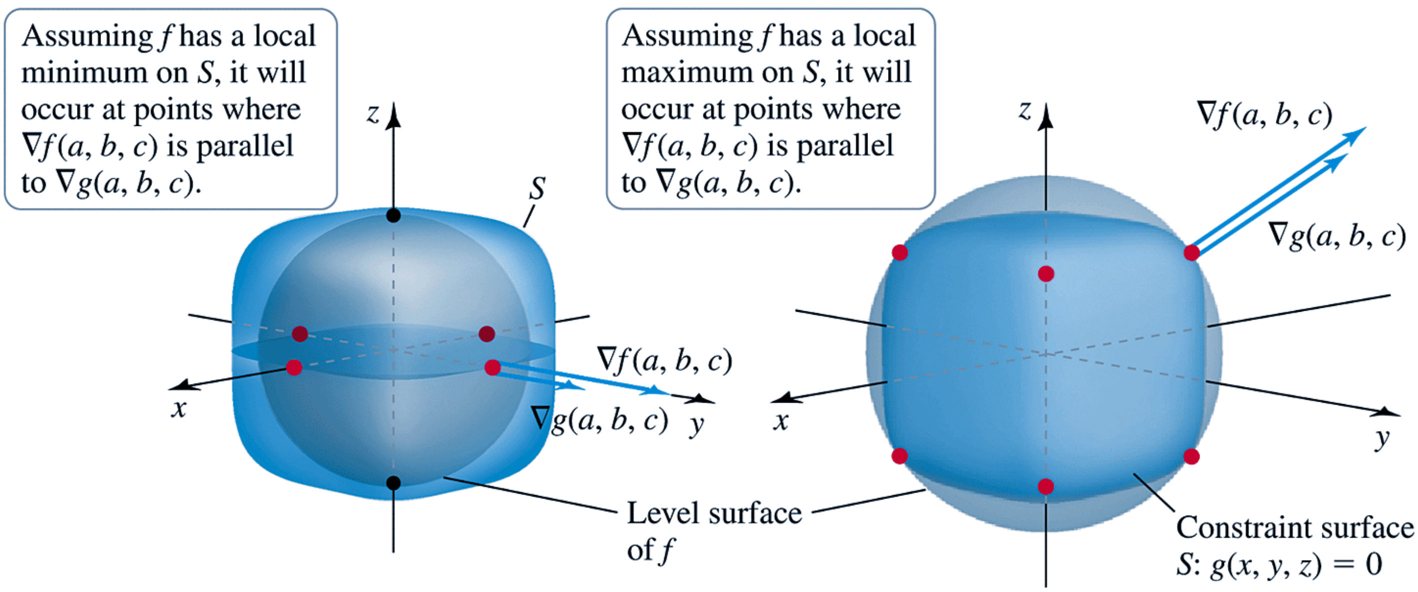
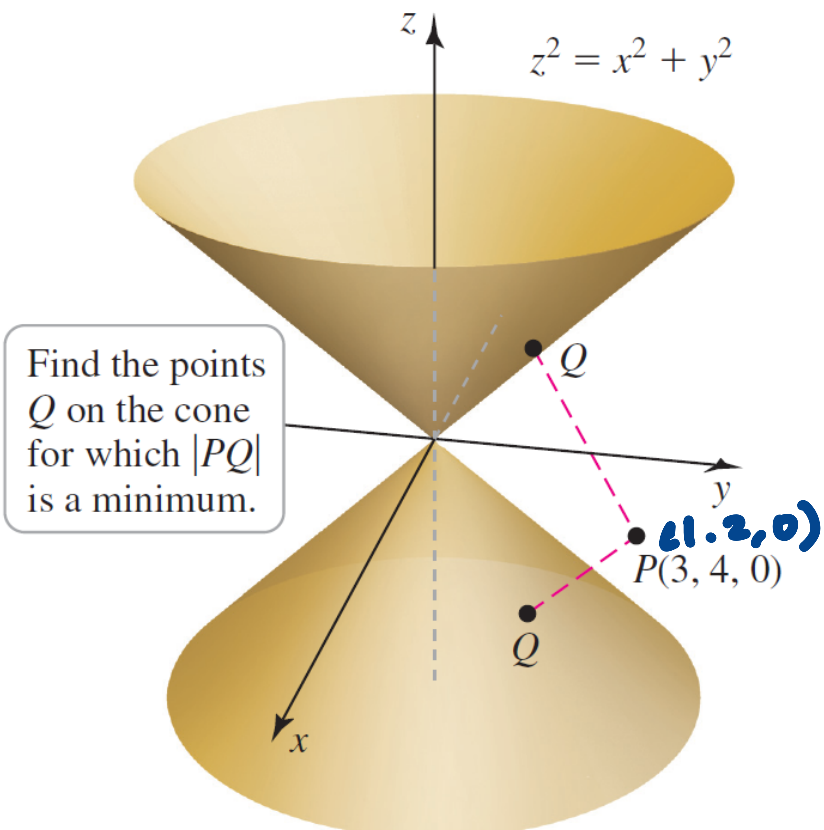


Figure 15.83


z
$z^2 = x^2 + y^2$
Find the points Q on the cone for which $|PQ|$ is a minimum.
Q
y
(1.2,0)
P(3, 4, 0)
Q
x

Example - Cloest point

Find the point on the cone $z^2 = x^2+y^2$ that is closest to the point $(1, 2, 0)$

Minimize $\sqrt{(x-1)^2+(y-2)^2+z^2}$ subject to $z^2=x^2+y$

Minimize $\underbrace{(x-1)^2+(y-2)^2+z^2}_{f(x,y,z)}$ subject to $\underbrace{z^2-x^2-y^2}_{g(x,y,z)}=0$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \begin{aligned} 2(x-1) &= -2\lambda x \\ 2(y-2) &= -2\lambda y \\ 2z &= 2\lambda z \end{aligned} \Rightarrow z=0 \text{ or } \lambda = 1.$$

$z=0$: $0^2 = x^2+y^2 \Rightarrow 0 = x^2+y^2 \Rightarrow x=0$ and $y=0$ pt: $(0, 0, 0)$

$\lambda = 1$: $2(x-1) = -2x \Rightarrow 4x = 2 \Rightarrow x = \frac{1}{2}$ pt: $(\frac{1}{2}, 1, \frac{\sqrt{5}}{2}), (\frac{1}{2}, 1, -\frac{\sqrt{5}}{2})$

$2(y-2) = -2y \Rightarrow 4y = 4 \Rightarrow y = 1$ $z^2 = (\frac{1}{2})^2 + 1^2 = \frac{1}{4} + 1 = \frac{5}{4}$

pts	$f = (x-1)^2 + (y-2)^2 + z^2$
$(0, 0, 0)$	$(0-1)^2 + (0-2)^2 + 0^2 = 5$ ← max
$(\frac{1}{2}, 1, \frac{\sqrt{5}}{2})$	$(\frac{1}{2}-1)^2 + (1-2)^2 + (\frac{\sqrt{5}}{2})^2 = \frac{1}{4} + 1 + \frac{5}{4} = \frac{10}{4} = \frac{5}{2}$ → min
$(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2})$	$(\frac{1}{2}-1)^2 + (1-2)^2 + (-\frac{\sqrt{5}}{2})^2 = \frac{1}{4} + 1 + \frac{5}{4} = \frac{10}{4} = \frac{5}{2}$ → min

closest pts: $(\frac{1}{2}, 1, \frac{\sqrt{5}}{2})$ and $(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2})$